Polarity in DEA models

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Abstract

This article discusses models in data envelopment analysis (DEA) relaxing the standard convexity assumptions. The basic model treats mutually incomparable pairs of sets to be generated by a procedure proposed earlier. Each pair consists of a consumption set and a production set of feasible input-output combinations. Two fundamental operations by the procedure are based on intersection and convex hull generation in the input-output space. A polarity analysis is performed which, subject to the usual assumptions about free disposability and nonnegativity, appears fruitful to do in the framework of blocking and antiblocking sets. It is shown how this leads to an interchange of the above operations extending some classical results from convex analysis. The last part of the paper presents a pair of linear programming models calculating a Farrell productivity index based on a preceeding application of the procedure. This is a generalization of the classical linear programming models in DEA subject to standard assumptions about convexity.

Keywords: Data envelopment analysis, polarity.

1 Introduction

The models considered in Data Envelopment Analysis compares the performance of decision making units subject to some assumptions about the possibility space of input-output vectors. The comparison is done under various assumptions, in particular about convexity and free disposability. The original models by Banker [2] and Banker, Charnes and Cooper [3] operates with free disposability and convexity of the entire production possibility space. They have been widely applied. The model proposed by Tulkens [19] operates only with free disposabiblity, such that comparison is only possible by domination. Petersen proposed in [13] a relaxation requiring convexity of the projections into input and output space, respectively. Further work along this line has been done by Bogetoft [6]. Consider the set of mutually non-dominating input-output vectors. For any subset create the convex polytope spanned by the subset, but keep only polytopes that do not dominate any input-output vector. Kuosmanen [12] constructs the smallest (usually nonconvex) polyhedron containing all remaining polytopes and adds a cone to ensure free disposability. Post [14] considers a convex transformation of a non-convex possibility set by means of so-called transconvex functions.

Bogetoft et al. [7] propose a method which subject to free disposability constructs the smallest possibility set having convex projections in input and output space. It operates with a selection of pairs of input-output vectors such that any vector of a pair constitutes a feasible input-output combination. The pairs are formed by a sequence of alternating intersection and convex hull operations. For a given input vector the convex projection in the output space is obtained as the largest output part from all pairs containing the input vector. By construction it is the production set consisting of all outputs that may be obtained by the given input. Symmetrically one may also construct the consumption set consisting of all inputs from which a given output vector may be reached.

The present analysis gives a dual description of the production and consumption sets. The blocking and antiblocking theory, originally developed by Fulkerson [10] for the analysis of certain problems in extremal combinatorics, appears to offer an appropriate framework. This is due to the assumptions about nonnegativity and free disposability which imply that a consumption set undertakes the form of a blocking polyhedron and a production set the form of an antiblocking polyhedron. This establishes an involutory polarity which gives additional flexibility by enabling an interchange of the above intersection and convex hull operations. The method generates pairs of convex polyhedra. However, the duality analysis to be carried out here considers not just polyhedra but investigates the larger class of convex sets. This is in agreement with the setting selected in previous similar studies on aureoled sets of Ruys and Weddepohl [16] and Weddepohl [20] and on generalized blockers and antiblockers by Tind [18]. In this context we should also mention the book by Färe and Primont [9] in which a basic duality theory is discussed for convex production models which do not necessarily have a polyhedral structure. The main distinction between the present and previous work is the particular analysis performed here as a result of the assumption about nonnegativity together with free disposability.

In DEA the efficiency of a specific decision making unit may be measured by a productivity index which subject to the usual convexity assumptions is calculated by linear programming. An extended linear programming model is proposed for the calculation of a productivity index subject to the present relaxed convexity assumptions. It is finally demonstrated how a classical calculation of an index may be derived as a special case of the model together with a presentation of a simple linear programming model for calculation of the index suggested by Tulkens [19].

The next section gives an introduction to the method generating convex consumption and production sets. Section 3 discusses the polarity theory for antiblocking and blocking sets and is subsequently applied in a DEA context in Section 4. Section 5 formulates the linear programming model for the calculation of the productivity index under the relaxed convexity assumptions.

2 Convexity in DEA

Consider a set U of decision making units, DMU's, and for each $i \in U$ let x_i denote the nonnegative input vector and y_i the nonnegative output vector of the i'th DMU. This means that x_i can produce y_i, or in other words that the input-output pair (x_i, y_i) represents a feasible input-output combination. Let x and y denote arbitrary elements of the input- and output space, respectively. Additional combinations (x, y) may naturally be feasible, dependent on some classical assumptions to be discussed below.

Let $L_i = \{x \mid x \geq x_i\}$ and $P_i = \{y \geq 0 \mid y \leq y_i\}$ for $i \in U$. So, geometrically L_i is a translation of the nonnegative orthant of the input space. In the output space P_i is a box, which has full dimension if and only if all elements in y_i are strictly positive. See Figure 1 for an illustration. DEA models usually assume *free disposability.* With the current notation this may be expressed by requiring all points $(x, y) \in (L_i, P_i)$ to be feasible

Figure 1: Feasible input-output combinations for i'th DMU.

input-output combinations.

Let T denote the production possibility set which by definition consists of all feasible input-output combinations. If free disposability is the only assumption made we are dealing with the so-called FDH (free disposable hull) model introduced by Tulkens [19]. For the FDH model we get

$$T = \cup_{i \in U}(L_i, P_i). \tag{1}$$

The classical DEA models additionally introduce some *convexity* conditions. If

$$T = \text{ conv } \{\cup_{i \in U}(L_i, P_i)\} \tag{2}$$

we get the varying return to scale (vrs) model developed in Banker [2] and Banker, Charnes and Cooper [3].

Let L_0 be the non-negative orthant of the input space and let P_0 be the zero vector in the output space. If this pair (L_0, P_0) is included, i. e. $0 \in U$ we implicitly say that any input may produce no output. In that case condition (2) gives us the decreasing return to scale (drs) model considered by the same authors.

For an arbitrary set S let cone S denote the *convex cone* generated by conv S, i. e. cone $S = \{\lambda s \mid s \in \text{ conv } S \text{ and } \lambda \in \mathbb{R}_+\}$. If

$$T = \text{ cone } \{\cup_{i \in U}(L_i, P_i)\} \tag{3}$$

we get the constant return to scale (crs) model proposed by Charnes, Cooper and Rhodes [5].

We shall next consider the projections of the possibility set into the input and output space, respectively. For a given input vector x define the *production set* by $P(x) = \{y \mid (x, y) \in T\}$ i. e. $P(x)$ consists of all possible outputs for the input vector x. Similarly, for a given output vector y let $L(y)$ denote the *consumption set* defined by $L(y) = \{x \mid (x, y) \in T\}$.

If T is convex, which is the case for the models satisfying (2) and (3), then the projections $P(x)$ and $L(y)$ are also convex. However, if T is non-convex as in the FDH-model from (1) then $P(x)$ and $L(y)$ are generally non-convex too.

A basic question is: What is the minimal possibility set T satisfying free disposability such that the production set $P(x)$ and the consumption set $L(y)$ are convex for all $(x, y) \in T$? To emphasize the symmetry of the framework let $\uplus$ denote the convex hull opration and define the following two pairs of sets for arbitrary indices $i, j \in U$:

$$(L_n, P_n) = (L_i \cap L_j, P_i \uplus P_j) \tag{4}$$

and

$$(L_m, P_m) = (L_i \uplus L_j, P_i \cap P_j). \tag{5}$$

Due to free diposability any $x \in L_i \cap L_j$ and $y \in P_i \cup P_j$ constitute a feasible input-output combination. Since the the production set $P(x)$ is required to be convex this is also true for any pair $(x, y) \in (L_n, P_n)$. Similarly any pair $(x, y) \in (L_m, P_m)$ is a feasible combination, since the consumption set $L(y)$ is convex.

Hence, (4) and (5) generate pairs of sets consisting of feasible input-output combinations in addition to the original pairs (L_i, P_i) for $i \in U$. This approach is continued in an iterative procedure to be outlined below.

Next introduce the concept of *dominance*. If $L_i \subseteq L_j$ and $P_i \subseteq P_j$ for some $i, j \in U$ then the pair (L_j, P_j) is said to *dominate* the pair (L_i, P_i). In this case all input-output combinations of the i' th pair are included in the j'th pair and index j may be removed.

This may be formalised by the following

Procedure

Start: Let $N_1 = U$ and l = 1.

Step 1: Create by (4) and (5) all new pairs (L_n, P_n) and (L_m, P_m) based on existing pairs with indices $i, j \in N_l$.

Step 2: Let N_{l+1} consist of all pairs in N_l together with the new ones created in Step 1.

Step 3: Remove from N_{l+1} any pair which is dominated by another pair in N_{l+1}.

Step 4: If $N_{l+1} = N_l$ no more non-dominated pairs can be generated and the procedure terminates. Otherwise let $l := l + 1$ and go to Step 1.

The procedure creates a series of indices. This series to be denoted by $\mathcal{N}$ may be finite or infinite dependent on possible fulfillment of the termination criterion in Step 4. Consider the possibility set T defined by

$$T = \{(x, y) \mid \exists i \in \mathcal{N} \text{ such that } (x, y) \in (L_i, P_i)\}.$$

By construction this is the smallest possible possibility set satisfying the desired properties: It contains the input-output points of the DMU's, satisfies free disposability and has convex projections. More details including conditions for finiteness of the procedure may be found in Bogetoft et al. [7].

3 Non-negative polarity

The previous section concentrated on the development of pairs of feasible input-output combinations of a polyhedral structure. In this section, however, we generalize to sets that are not necessarily polyhedral to conform with the standard assumptions for production and consumptions sets discussed in microeconomic theory. We treat those sets in a duality perspective in which the input and output space are going to be considered independently. First let us consider polarity properties of the output space.

3.1 Antiblockers

Let p denote the dimension of the output space and introduce a set $P \in \mathbb{R}^p$ satisfying

$$[(\text{cl conv } P) - \mathbb{R}^p_+] \cap \mathbb{R}^p_+ = P. \tag{6}$$

Any output part P_i of an input-output combination generated by the previous procedure is a polyhedron satisfying this property. By (6) it follows directly that P is convex and nonnegative. Moreover $0 \in P$ unless P is empty. Hence we assume that P is nonempty. The next proposition together with (6) immediately shows that P is also closed.

Proposition 1 (cl conv P) $- \mathbb{R}^p_+$ *is closed.*

Proof: The proof is based on a result stated in Rockafellar [15, Theorem 20.3] about the addition of a closed convex set C_1 and a polyhedral convex set C_2. The result says that $C_1 + C_2$ is closed provided that every direction of recession of C_1 whose opposite is a direction of recession of C_2 is actually a direction in which C_2 is linear. Therefore our proposition is valid if these properties hold for $C_1 =$ cl conv P and $C_2 = -\mathbb{R}^p_+$. Let d be a direction of recession of cl conv P. Then $d \in \mathbb{R}^p_+$ since cl conv P is nonnegative. Hence d is the opposite of a direction of recession in $-\mathbb{R}^p_+$. So the proof is finished if we can show that d recedes in a direction in which cl conv P is linear. Consider the subspace of $\mathbb{R}^p_+$ correponding to the positive components of d. By (6) the nonnegative orthant of this subspace is contained in P and hence in cl conv P. This means that the projection of cl conv P into the subspace is equal to the nonnegative orthant which is linear in any nonnegative direction. Hence the projection of d is a direction in which the projection of cl conv P is linear. Since cl conv P in nonnegative and contains $0 \in \mathbb{R}^p_+$ we get that d itself is a direction in which the entire cl conv P is linear. ■

The above analysis can be summarized into the statement that condition (6) holds if and only if P satisfies the following properties:

- P is a non-negative, closed and convex.
- P satisfies free disposability in the output space, i. e. $0 \leq \tilde{y} \leq y$ and $y \in P$ imply $\tilde{y} \in P$.

These properties are assumptions commonly made in the classical literature on microeconomics, see e. g. Debreu [8].

From a dual perspective a set satisfying (6) is naturally studied in the framework of so-called antiblockers. This framework was introduced for polyhedra by Fulkerson [11] for the investigation of polarity properties of some problems in extremal combinatorics. Here we shall see that the basic concept appears useful for an investigation of similar properties in data envelopment analysis.

Antiblockers are similar to the notion of polar sets, see Rockafellar [15], except that all elements to be considered are nonnegative, which is the case in the DEA framework too. The familiarity with polar sets means that the following propositions may be derived from similar propositions for polar sets while taking appropriate consideration to nonnegativity. However to keep the paper selfcontained and in order to establish the connection to blockers to be considered in Section 3.2 we include direct proofs:

Formally an antiblocker of P is defined by

$$\mathcal{A}(P) = \{y^* \in \mathbb{R}^p_+ \mid y^*y \leq 1 \text{ for all } y \in P\}. \tag{7}$$

Note first that $\mathcal{A}(P)$ is nonempty and that it by replacing P satisfies condition (6). Introduce an additional nonempty set O satisfying (6) and consider the following

Proposition 2 $\mathcal{A}(P \uplus O) = \mathcal{A}(P) \cap \mathcal{A}(O)$.

Proof: Since $P \uplus O \supseteq P$ we have that $\mathcal{A}(P \uplus O) \subseteq \mathcal{A}(P)$. Similarly we obtain that $\mathcal{A}(P \uplus O) \subseteq \mathcal{A}(O)$ showing that

$$\mathcal{A}(P \uplus O) \subseteq \mathcal{A}(P) \cap \mathcal{A}(O).$$

To prove the converse inclusion let $y^* \in \mathcal{A}(P) \cap \mathcal{A}(O)$. Then $y^*y \leq 1$ for all $y \in P$ and similarly $y^*y \leq 1$ for all $y \in O$. This shows that $y^*y \leq 1$ for all

$y \in P \cup O$ and hence also for all convex combinations of elements in $P \cup O$. Thus $y^*y \leq 1$ for all $y \in P \uplus O$ or equivalently $y^* \in \mathcal{A}(P \uplus O)$. ■

The next proposition states the involutory property for antiblockers. This was stated by Fulkerson [10] for polyhedra. A generalized version appears in Tind [18]. We give a direct proof here under the present assumptions.

Proposition 3 $\mathcal{A}(\mathcal{A}(P)) = P$.

Proof: Obviously $\mathcal{A}(\mathcal{A}(P)) \supseteq P$. By (6) and Proposition 1 $P - \mathbb{R}^p_+$ contains 0 and is closed. Moreover, the recession cone of $P - \mathbb{R}^p_+$ includes the nonpositive orthant $\mathbb{R}^p_-$. Hence $P - \mathbb{R}^p_+$ may be described as the intersection of closed halfspaces of the form $\{y \in R^p_+ \mid y^*y \leq 1\}$, where $y^* \in \mathbb{R}^p_+$. Assume that $y \in \mathbb{R}^p_+$ and $y \notin P$. Then by (6) also $y \notin P - \mathbb{R}^p_+$. Hence by the theorem of separating hyperplanes a nonnegative normal y^* exists such that $y^*y \leq 1$ for all $y \in P - \mathbb{R}^p_+$, implying that $y^* \in \mathcal{A}(P)$, while $y^*y > 1$. This shows that $y \notin \mathcal{A}(\mathcal{A}(P))$. ■

As a counterpartner to Proposition 2 we shall state the following dual version.

Proposition 4 $\mathcal{A}(P \cap O) = \text{cl } (\mathcal{A}(P) \uplus \mathcal{A}(O))$.

Proof: Since $\mathcal{A}(P)$ and $\mathcal{A}(O)$ satisfy condition (6) we may use Proposition 2 to obtain

$$\mathcal{A}(\mathcal{A}(P) \uplus \mathcal{A}(O)) = \mathcal{A}(\mathcal{A}(P)) \cap \mathcal{A}(\mathcal{A}(O)).$$

By Proposition 3 we get

$$\mathcal{A}(\mathcal{A}(P) \uplus \mathcal{A}(O)) = P \cap O$$

and further

$$\mathcal{A}(\mathcal{A}(\mathcal{A}(P) \uplus \mathcal{A}(O))) = \mathcal{A}(P \cap O).$$

For any set $Y \in \mathbb{R}^p$ we have that $\mathcal{A}(Y) = \mathcal{A}(\text{cl } (Y))$. This implies that

$$\mathcal{A}(\mathcal{A}(\text{cl } (\mathcal{A}(P) \uplus \mathcal{A}(O)))) = \mathcal{A}(P \cap O).$$

Since the set cl $(\mathcal{A}(P) \uplus \mathcal{A}(O))$ satisfies condition (6) additional application of Proposition 3 implies the requested result. ■

The following example shows the neccessity of the closure operation cl in proposition 4 in general.

Example 1

Consider in the two dimensional output space the two polyhedra $P =$ conv $[(0,0),(0,0.5),(0.5,0)]$ and $O =$ conv $[(0,0),(1,0)]$. They each have a form that could be the output part of a feasible input-output pair. However O has no output in the first component. Now $\mathcal{A}(P) = \{(y_1^*, y_2^*) \mid (0,0) \leq (y_1^*, y_2^*) \leq (2,2)\}$ and $\mathcal{A}(O) = \{(y_1^*, y_2^*) \mid 0 \leq y_1^* \leq 1 \text{ and } y_2^* \geq 0\}$. In this case $\mathcal{A}(P) \uplus \mathcal{A}(O)$ is not closed. See Figure 2.

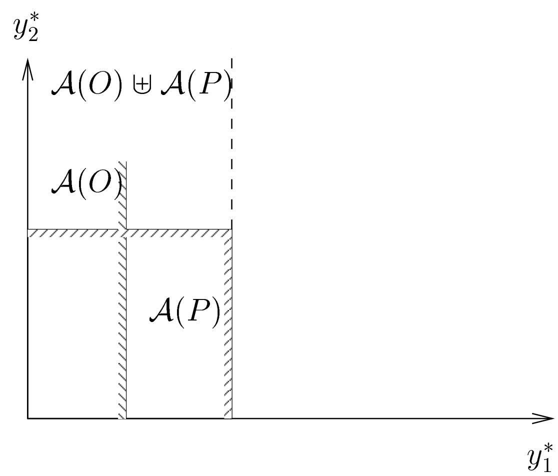

Figure 2: $\mathcal{A}(O) \uplus \mathcal{A}(P)$ is not closed.

The trouble in the above example is the lower dimension of O. Indeed, the closure operation can be removed if $\mathcal{A}(P)$ and $\mathcal{A}(O)$ have the same recession cone, see Rockafellar [15, Corollary 9.8.1]. This occurs if the cones generated by P and O have full dimension p in the output space, in which case $\mathcal{A}(P)$ and $\mathcal{A}(O)$ are bounded. These observations can be summarized into the following

Corollary 1 *If the output sets P and O are both full dimensional then* $\mathcal{A}(P \cap O) = \mathcal{A}(P) \uplus \mathcal{A}(O)$.

The above results make it possible to interchange the role of intersection and convex union as summarized in

Proposition 5

$$P \cap O = \mathcal{A}(\mathcal{A}(P) \uplus \mathcal{A}(O)) \tag{8}$$

$$\textit{and } \operatorname{cl}\,(P \uplus O) = \mathcal{A}(\mathcal{A}(P) \cap \mathcal{A}(O)). \tag{9}$$

Proof: $\mathcal{A}(P)$ and $\mathcal{A}(O)$ satisfy property (6). Hence by Proposition 2 we get $\mathcal{A}(\mathcal{A}(P) \uplus \mathcal{A}(O)) = \mathcal{A}\mathcal{A}(P) \cap \mathcal{A}\mathcal{A}(O)$. Finally, application of Proposition 3 on the last terms implies (8). (9) follows by Propositions 3 and 4.

3.2 Blockers

We shall here study the input sets in the framework of blockers. Let r denote the dimension of the input space and consider a nonempty set $L \in \mathbb{R}^r$. In this framework the condition (6) valid for antiblockers changes to

$$[(\text{cl conv } L) + \mathbb{R}^r_+] \cap \mathbb{R}^r_+ = L. \tag{10}$$

Since by (10) L and $\mathbb{R}^r_+$ have the same recession cone $\mathbb{R}^r_+$ it follows in this case by Rockafeller [15, Corollary 9.8.1] that (cl conv L) $+\mathbb{R}^r_+$ is closed. Hence also L is closed. In parallel with the description of antiblockers we thus here get that L satisfies condition (10) if and only if

- L is nonnegative, closed, convex and $0 \notin L$.
- L satisfies free disposability in the input space, i. e. $x \leq \tilde{x}$ and $x \in L$ imply $\tilde{x} \in L$.

The input sets L_j generated by the procedure in section 2 satisfy equation (10).

The definition of a blocker is similar to the definition (7) of an antiblocker, however with the inequality reversed. So, the the blocker $\mathcal{B}(L)$ of L is going to be defined by

$$\mathcal{B}(L) = \{x^* \in \mathbb{R}^r_+ \mid x^*x \geq 1 \text{ for all } x \in L\}. \tag{11}$$

The blocker is nonempty as $0 \notin L$ and satisfies (10) replacing L. Sets of similar structure have been studied as socalled aureoled sets by Weddepohl [20] and Ruys and Weddepohl [16] and as socalled reverse polar sets by Tind [18]. However, here we transfer the proofs from Section 3.1 while indicating some slight changes.

Introduce an additional set K satisfying (10). Similar to Proposition 2 we get

Proposition 6 $\mathcal{B}(L \uplus K) = \mathcal{B}(L) \cap \mathcal{B}(K)$.

Proof: This is similar to the proof of proposition 4, however with inequalities reversed. ■

With a similar proof as for Proposition 3 we have

Proposition 7 $\mathcal{B}(\mathcal{B}(L)) = L$.

We also get

Proposition 8 $\mathcal{B}(L \cap K) = \mathcal{B}(L) \uplus \mathcal{B}(K)$.

Proof: The proof is similar to the proof of Proposition 4 together with the observation that the closure operation is not needed since $\mathcal{B}(L)$ and $\mathcal{B}(K)$ have the same recession cone $\mathbb{R}^r_+$. ■

In analogue with Proposition 5 we may similarly state and prove

Proposition 9

$$\begin{aligned} L \cap K &= \mathcal{B}(\mathcal{B}(L) \uplus \mathcal{B}(K)) \\ \textit{and } L \uplus K &= \mathcal{B}(\mathcal{B}(L) \cap \mathcal{B}(K)). \end{aligned}$$

4 Polarity in DEA

Proposition 5 and Proposition 9 make it possible to substitute the convex union operation in (4) and (5) in the original spaces by an intersection operation in the dual spaces, respectively. Similarly the intersection operation in (4) and (5) may be substituted by a convex union operation in the dual spaces. In particular if the original output sets P_i for $i \in U$ are full dimensional then also all subsequent output sets generated by (4) and (5) become full dimensional. In this case we may use Corollary 1 and dismiss the closure operation in Proposition 5.

Example 2

Consider in the two dimensional output space two DMU's indexed by 1 and 2 and with output vectors $(5, 4)$ and $(6, 1)$, respectively. We thus have
$P_1 = \{(y_1, y_2) \mid (0, 0) \leq (y_1, y_2) \leq (5, 4)\}$
$= \{(y_1, y_2) \geq (0, 0) \mid \frac{1}{5}y_1 \leq 1, \frac{1}{4}y_2 \leq 1\}$ and
$P_2 = \{(y_1, y_2) \mid (0, 0) \leq (y_1, y_2) \leq (6, 1)\}$
$= \{(y_1, y_2) \geq (0, 0) \mid \frac{1}{6}y_1 \leq 1, y_2 \leq 1\}$.
These sets are incicated on Figure 3. We obtain that
$\mathcal{A}(P_1) = \{(y_1^*, y_2^*) \geq (0, 0) \mid 5y_1^* + 4y_2^* \leq 1\}$
and $\mathcal{A}(P_2) = \{(y_1^*, y_2^*) \geq (0, 0) \mid 6y_1^* + 1y_2^* \leq 1\}$.
Those sets are also indicated on Figure 3.
Then $\mathcal{A}(P_1) \cap \mathcal{A}(P_2) = \{(y_1^*, y_2^*) \geq (0, 0) \mid 5y_1^* + 4y_2^* \leq 1 \text{ and } 6y_1^* + 1y_2^* \leq 1\}$ implying that

$\mathcal{A}(\mathcal{A}(P_1) \cap \mathcal{A}(P_2))$
$= \{(y_1, y_2) \geq (0,0) \mid \frac{3}{19}y_1 + \frac{1}{19}y_2 \leq 1, \frac{1}{6}y_1 \leq 1 \text{ and } \frac{1}{4}y_2 \leq 1\}$
$= P_1 \uplus P_2$.
This shows that the antiblocker of $\mathcal{A}(P_1) \cap \mathcal{A}(P_2)$ is equal to $P_1 \uplus P_2$ illustrating (9). The two sets are shown on Figure 3 by thick borderlines.

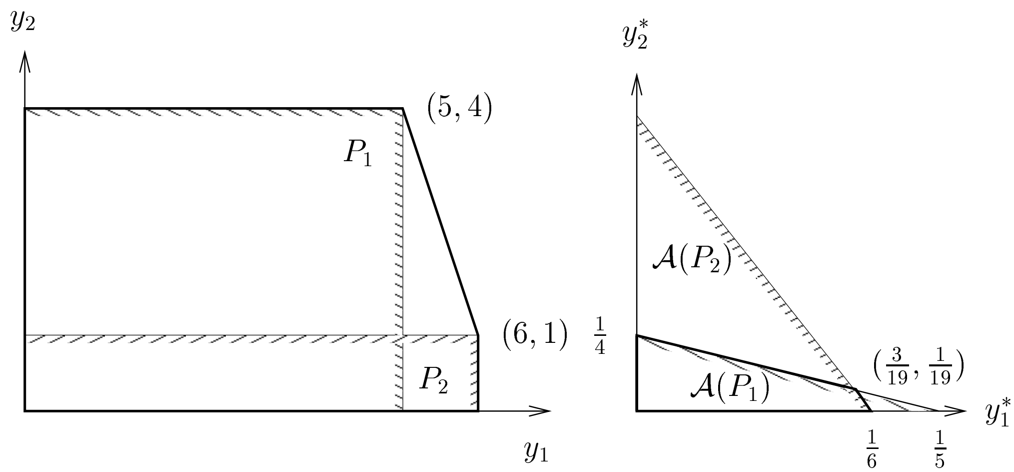

Figure 3: Example.

By definition we have for any $y^* \in \mathcal{A}(P)$ that $\{y \mid yy^* \leq 1\} \supseteq P$. In other words for fixed $y^* \in \mathcal{A}(P)$ then $yy^* \leq 1$ is a valid inequality for P. The inequality has nonnegative coefficients and a positive right hand side, normalized to one. The inequality may be interpreted as a resource constraint in a usual fashion. The coefficient vector y^* may be interpreted as a consumption rate of the resource per unit of the output components. We have of course special interest in the binding constraints as they dominate all other constraints with coefficients in $\mathcal{A}(P)$. This set is finite and constitutes together with possible facets of the nonnegative orthant all facets of the polyhedron $\mathcal{A}(P)$. A similar discussion may be done also for blockers. More discussion about the polyhedral structure of the sets generated is done in the next Section 5.

5 Productivity index

This section discusses how a productivity index may be introduced under relaxed convexity.

The input sets L_i indexed by $\mathcal{N}$ may be partially ordered with respect to inclusion. Similarly for the output sets P_i. If the procedure described in Section 2 has finite termination then the order of the sets in the input space and the order of the corresponding sets in the output sets are mutually reverse. The two orderings constitute a pair of dual lattices with respect to intersection and convex union. For details see Bogetoft et al. [6]

In this context it is possible to introduce an efficiency score of a particular DMU with respect to all DMU's. Let $(x, y) \in \mathbb{R}^{r \times p}$ be an input-output vector of the selected DMU. Consider a pair (L_i, P_i) with $i \in \mathcal{N}$. In accordance with the tradition we say that a point $x \in L_i$ is *efficient* in L_i if no other point $\hat{x} \in L_i$ exists with $\hat{x} \leq x$ and $x \neq \hat{x}$. Similarly, a point $y \in P_i$ is *efficient* in P_i if no other point $\hat{y} \in P_i$ exists with $\hat{y} \geq y$ and $y \neq \hat{y}$.

Assume that the procedure in Section 2 has terminated after a finite number of steps. If a common index $i \in \mathcal{N}$ exists such that x_0 is efficient in L_i and y_0 efficient in P_i we then say that the DMU with the input-output vector (x_0, y_0) is *efficient* with respect to all DMUs. Otherwise, components of x_0 may be diminished or components of y_0 increased. This question shall be examined through possible scaling. Traditionally DEA operates with an input oriented efficiency score as well as an output oriented efficiency score by means of a Farrell index. In the current setting the input oriented efficiency score is the optimal value of the program

$$\min\{\theta \mid \exists i \in \mathcal{N} \text{ where } (\theta x_0, y_0) \in (L_i, P_i)\}. \tag{12}$$

Due to the lattice structure the calculations in (12) select in the input space the largest set L_i such that $(x_0, y_0) \in (L_i, P_i)$. Alternatively, due to the dual lattice structure the same pair of sets is found by selecting the smallest set P_i such that $(x_0, y_0) \in (L_i, P_i)$.

All sets generated by the procedure are polyhedral. By the blocking theory developed in Section 3.2 we may therefore rewrite an input set L_i as

$$L_i = \{x \geq 0 \mid B_i x \geq \mathbf{1}\} \tag{13}$$

where B_i is a nonnegative matrix and $\mathbf{1}$ is a vector of ones of conformable dimensions. Similarly by Section 3.1 we have

$$P_i = \{y \geq 0 \mid A_i y \leq \mathbf{1}\} \tag{14}$$

where A_i is a nonnegative matrix and $\mathbf{1}$ is a vector of ones of conformable dimensions.

For a given index i such that $y_0 \in P_i$ the program (12) can be transformed into the following linear programming problem.

$$\begin{array}{ll} \min\limits_{\theta} & \theta \\ \text{s.t.} & B_i x_0 \theta \geq \mathbf{1}. \end{array}$$

The objective is to select an index i giving the minimal value of the above program. This can be formulated as a disjunctive programming problem leading further to a linear programming formulation, see Balas [1]. Using this technique we introduce the additional variables $z_i \in \mathbb{R}_+$ and $\theta_i \in \mathbb{R}$ for $i \in \mathcal{N}$ and consider the linear programming problem.

$$\begin{array}{ll}
\min\limits_{\theta_i, z_i} & \sum\limits_{i \in \mathcal{N}} \theta_i \\
\text{s.t.} & B_i x_0 \theta_i - z_i \mathbf{1} \geq 0 \text{ for all } i \in \mathcal{N} \\
 & (A_i y_0 - \mathbf{1}) z_i \leq 0 \text{ for all } i \in \mathcal{N} \\
 & \sum\limits_{i \in \mathcal{N}} z_i = 1 \\
 & z_i \geq 0.
\end{array}$$

This program is linearly homogeneous in the the z_i variables, $0 \leq z_i \leq 1$. Hence an optimal solution may be found by putting a single variable z_i equal to 1 and the remaining ones to 0. The selected value will correspond to the input-output pair (L_i, P_i) in (12) giving the minimal value of θ. It should be remarked however, that some work is required for the determination of the matrices A_i and B_i, in particular if the index set $\mathcal{N}$ is large. This size depends on the termination performance of the procedure for the considered instance. Perhaps a premature interruption gives a sufficient approximation of the possibility set for the determination of an efficiency score.

As an alternative to the closed halfspace characterization used in (13) and (14) the input and output sets may be characterized by their extreme points. For this purpose let S_i denote the index set of all extreme points of the input set L_i and let e_{ij} denote an extreme point, $j \in S_i$. Similarly, for the corresponding output set P_i let f_{ij} denote an extreme point together with the index set T_i. In the case of output sets some extreme points may be removed as they may be dominated by other extreme points with larger elements. This is due to free disposability in the output space where dominated extreme points may occur on the axes of coordinates. By introduction of the variables λ_{ij} and μ_{ij} we have

$$\begin{array}{rcl}
L_i & = & \{x \geq 0 \mid x \geq \sum\limits_{j \in S_i} e_{ij} \lambda_{ij}, \sum\limits_{j \in S_i} \lambda_{ij} = 1, \lambda_{ij} \geq 0\} \text{ and} \\
P_i & = & \{y \geq 0 \mid y \leq \sum\limits_{j \in T_i} f_{ij} \mu_{ij}, \sum\limits_{j \in S_i} \mu_{ij} = 1, \mu_{ij} \geq 0\}.
\end{array}$$

(Strictly speaking the non-negativity condition in L_i is not required as all extreme points e_{ij} are nonnegative in our case). In this framework the Farrell index in (12) may be calculated by the following linear program.

$$\min \quad \sum_{i \in \mathcal{N}} \theta_i$$

$$
\begin{aligned}
\text{s.t.} \quad & \sum_{j \in S_i} \lambda_{ij} e_{ij} \leq x_0 \theta_i \text{ for all } i \\
& \sum_{j \in T_{ij}} \mu_{ij} f_{ij} \geq y_0 z_i \text{ for all } i \\
& \sum_{j \in S_{ij}} \lambda_{ij} = z_i \text{ for all } i \\
& \sum_{j \in T_{ij}} \mu_{ij} = z_i \text{ for all } i \\
& \sum_{i \in \mathcal{N}} z_i = 1 \\
& \lambda_{ij}, \mu_{ij} \geq 0 \text{ for all } i, j \\
& z_i \geq 0 \text{ for all } i.
\end{aligned}
$$

It should be noted that the extreme points of L_i are all non-negative normals of facets for the antiblocker $\mathcal{A}(L_i)$. By the involutory correspondence stated in Proposition 3 we symmetrically have that the extreme points of $\mathcal{A}(L_i)$ correspond to the non-negative facets of L_i, which again is the minimal set of rows in B_i required to define L_i by (14). For details see Fulkerson [10].

We shall see that the above linear programming model is a generalization of some classical models as well.

With only a single input-output pair, i. e. when $|\mathcal{N}| = 1$, we may delete index i and denote the single pair by (L, P). Assume additionally that the number of non-dominated extreme points are the same in the two sets L and P. Denote this number by S. Furthermore, let the vector λ be equal to μ. In this setting we get the classical varying return to scale model studied in (2), in which the extreme points correspond to the decision making units. With the current notation we obtain the usual standard form:

$$
\begin{aligned}
\min \quad & \theta \\
\text{s.t.} \quad & \sum_{j \in S} \lambda_j e_j \leq x_0 \theta \\
& \sum_{j \in S} \lambda_j f_j \geq y_0 \\
& \sum_{j \in S} \lambda_j = 1 \\
& \lambda_j \geq 0.
\end{aligned}
$$

If the input-output pairs (L_i, P_i) are given by the original DMU's illustrated by Figure 1 then each set of the pair has only one non-dominated extreme point. We may thus remove the index j and additionally assume that $\lambda_i = \mu_i = z_i$. In this setting the model reduces to the FDH model studied in (1) and here stated as a linear programming problem,

$$\begin{array}{ll}
\min & \sum_{i \in \mathcal{N}} \theta_i \\
\text{s.t.} & z_i e_i \leq x_0 \theta_i \text{ for all } i \\
& z_i f_i \geq y_0 z_i \text{ for all } i \\
& \sum_{i \in \mathcal{N}} z_i = 1 \\
& z_i \geq 0 \text{ for all } i
\end{array}$$

in which $\mathcal{N}$ is the index set for the DMU's.

In all the above models we have for simplicity excluded the introduction of slacks to indicate the cases in which an input-output vector is efficient according to the index, i. e. $\theta = 1$, but nevertheless is dominated. Those slacks may however easily be introduced in a traditional manner, see for example Charnes et al. [4].

A similar analysis as above can be done in connection with the establishment an output oriented efficiency score.

6 Conclusion

This paper is primarily concerned with two issues.

One issue is related to a procedure for the generation of the smallest possibility set with convex projections, which is a relaxation of traditional convexity assumptions in DEA. During the procedure corresponding pairs of input and outputs sets are generated. The procedure is based on an alternating sequence of intersections and convex hull generations so that intersection of sets in the input space is performed together with generation of the convex hull of the corresponding sets in the output space, and vice versa. A polarity analysis has been performed allowing for an interchange of intersection and convex hull operation where this may be appropriate. This interchange is made via the dual space of the input and output space, respectively.

The other issue deals with the calculation of a Farrell index under the relaxed convexity conditions introduced. This leads to a linear programming formulation that is based on a preceeding application of the procedure. A crucial thing is how to generate all corresponding pairs. Addition of new pairs may lead to an improvement of the productivity index for a given input-output vector. However this improvement is believed to be very little once a certain number of pairs have been generated. So for practical purposes a premature interuption of the procedure is expected to have minor influence for the calculation of a sufficiently accurate index.

Other relaxations are possible. For example one may in a similar way analyze the free replicability model based on Tulkens [19]. This is an integer programming model, for which the duality theory for integer programming should be used, see for example Schrijver [17]. This implies that a polarity analysis may be done in a dual space consisting of Chvatal functions. Indeed the above analysis may be performed on any optimization model in DEA, for which an appropriate duality theory exists with no duality gap.

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